Steering in spin tomographic probability representation

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The steering property known for two-qubit state in terms of specific inequalities for the correlation function is translated for the state of qudit with the spin $j = 3/2$. Since most steering detection inequalities are based on the correlation functions we introduce analogs of such functions for the single qudit systems. The tomographic probability representation for the qudit states is applied. The connection between the correlation function in the two-qubit system and the single qudit is presented in an integral form with an intertwining kernel calculated explicitly in tomographic probability terms.

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I. INTRODUCTION

The problem of the quantum steering was introduced by E. Schrödinger in [1] as an answer to the paper of A. Einstein, B. Podolsky and N. Rosen [2] to generalize the EPR paradox. Steering reflects the kind of the quantum correlations which exist in the composite quantum systems. There are different kinds of the known quantum correlations as entanglement also available in the composite systems [1]. The quantum correlations available in the non composite systems are reflected by the phenomenon of the non contextuality [3]. Since the EPR steering can be applied in one-way quantum cryptography [4, 5] or in visualization of the two-qubit state tomography [6] the problem is discussed in a large number of recent papers.

There are many definitions of the term 'steering'. In [7] the EPR steering was defined as a form of a nonlocality in quantum mechanics, that is in between the entanglement and the Bell nonlocality. In [8] the notion of steering was reformulated. The EPR steering was considered as the ability of the first system to affect the state of the second system through the choice of the first systems measurement basis. Hence, the concept of the quantum steering can be introduced not only for the multipartite (joint) systems, but also for all systems (including non composite ones) with correlations [9].

The EPR steering can be detected through the violation of steering inequalities [5, 10–12]. Most of the steering inequalities are connected with the notion of the correlation function [10]. The aim of our paper is to demonstrate that the phenomenon of the quantum steering exists not only in the composite systems but also in the non composite systems like the single qudit state (e.g. the single qudit with the spin $j = 3/2$ or the four-level atom). Therefore, the main focus of our paper is the correlation function as characteristics of the steering phenomenon for the two-qubit and for the single qudit systems. To clarify the connection of the steering phenomenon in the composite and non composite systems we analyze the observables (Hermitian operators [13]) corresponding to the single qudit and the multi qudit systems. We show that there exist in the single qudit system the observables with the properties mathematically identical to the other the observables available in the multi qudit system. In this context we map the states of the single qudit systems to the states of the artificial multi qudit systems [14]. To characterize degrees of quantum correlations in the systems we use the tomographic probability representation of quantum mechanics [15]. We find the tomographic representation of the correlation functions that characterize the steering in the qunatum system. We introduce the connection between tomograms for the two-qubit system and the tomogram for the single qudit with the spin $j = 3/2$. To introduce the correlation function we consider the specific observable which is a complete analog of the observable used in the two-qubit system but studied in the single qudit with the spin $j = 3/2$ picture. Hence, we can introduce the notion of the steering and detect the steering phenomenon in the system without subsystems. The physical application of the steering phenomena in the single qudit with the spin $j = 3/2$ and for the four-level atom can be performed in the study of the information and the entropic properties of the superconducting multilevel circuits [16, 17] where the notion of the artificial two-level atoms playing the role of the qubits is used [18, 19].

The paper is organized as follows. In Sec. II we rewrite the correlation function for the two-qubit system in terms of the spin tomogram. The connection of the two partite system tomogram and the single qudit state tomogram is introduced in Sec. III. In Sec. IV the correlation function is rewritten by means of the tomogram and the

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notion of the EPR steering is extended to the case of the system without subsystems (a single qudit). The physical meaning of the correlation function in the single qudit systems is given. The prospectives and conclusions are given in Sec. V.

II. THE TWO-QUBIT STEERING IN THE SPIN TOMOGRAPHIC REPRESENTATION

The notion of the EPR steering is best studied on the example of the two-qubit state, where each qubit is defined on two dimensional Hilbert space. Let us define the two-qubit quantum system on a the Hilbert space. The density matrix of the system state in a four-dimensional Hilbert space $\mathcal{H}_{AB}$ is the matrix ρ_{AB} of the size 4×4 with nonnegative eigenvalues and $\rho_{AB}=\rho_{AB}^{\dagger}$, $Tr\rho_{AB}=1$ hold. The correlations in such system ρ_{AB} can be described by the joint probability function

$$P(a,b|A,B)=\int p_\lambda P(a|A,\lambda)P(b|B,\lambda)d\lambda, \qquad (1)$$

where $P(a|A,\lambda)$ is the probability distribution of the measurement outcomes a under setting A for a hidden variable λ. The hidden variable has the probability distribution p_λ and its hidden state is ρ_λ (a local hidden state (LHS)). If the following model of the correlation

$$P(a,b|A,B)=\int p_\lambda P(a|A,\lambda)Tr(\widehat{\pi}(b|B)\rho_\lambda^{(b)})d\lambda \qquad (2)$$

does not exist, then the state is steerable [10]. Here $\widehat{\pi}(b|B)$ is the projection operator for an observable parameterized by the setting B and the $\rho_\lambda^{(b)}$ is some pure state of the system B. The EPR steering can be detected through the violation of the steering inequalities. The steering inequalities are mostly based on the notion of the correlation function. The quantum correlation function for the two-qubit state is determined by

$$E(\overrightarrow{k_1},\overrightarrow{k_2})=Tr(\overrightarrow{k_1}\cdot\overrightarrow{\sigma}\otimes\overrightarrow{k_2}\cdot\overrightarrow{\sigma}\rho), \qquad (3)$$

where $\overrightarrow{\sigma}$ is the vector built out of the Pauli matrix, $\overrightarrow{k_1}$, $\overrightarrow{k_2}$ are the unit Bloch vectors of the measurement directions equal to ±1.

The tomographic probability distribution of the spin states allows to describe the states determined by the density matrix ρ of the two qubits by means of the tomogram. By definition the spin tomogram

$$\omega(\mathbf{x})=\omega(m_1,m_2,u)=<m_1m_2|u\rho u^\dagger|m_1m_2>$$

is the probability to obtain $m_1=-j_1,-j_1+1,\ldots,j_1$, $m_2=-j_2,-j_2+1,\ldots,j_2$, $j_{1,2}=0,1/2,1\ldots$ as the spin projections on directions given by the unitary matrix u. Here we used the notation $\mathbf{x}=(m_1,m_2,u)$ and the matrix u is the unitary rotation matrix of the size $N\times N$, where $N=(2j_1+1)(2j_2+1)$ holds. If we choose the rotation matrix as the direct product of the two matrices $u=u_1\otimes u_2$ of irreducible representations of the $SU(2)$ - group, i.e.

$$u_j=\begin{pmatrix}\cos\frac{\theta_j}{2}e^{\frac{i(\varphi_j+\psi_j)}{2}} & \sin\frac{\theta_j}{2}e^{\frac{i(\varphi_j-\psi_j)}{2}}\\ -\sin\frac{\theta_j}{2}e^{\frac{i(\psi_j-\varphi_j)}{2}} & \cos\frac{\theta_j}{2}e^{\frac{-i(\varphi_j+\psi_j)}{2}}\end{pmatrix},\quad j=1,2$$

then the latter tomography can be written as

$$\omega(m_1,m_2,u_1,u_2)= \qquad (4)$$
$$=<m_1m_2|u_1\otimes u_2\rho u_1^\dagger\otimes u_2^\dagger|m_1m_2>.$$

The matrices u_1 and u_2 depend only on the Euler angles $\{\theta_i,\varphi_i,\psi_i\}, i\in\{1,2\}$ which determine the directions of quantization, e.g., points on the Bloch sphere. Hence, we use the following notations $u_1=u_1(\theta_1,\varphi_1,\psi_1)=u_1(\vec{n}_1)$, $u_2=u_2(\theta_2,\varphi_2,\psi_2)=u_2(\vec{n}_2)$, where $\vec{n}_1$ and $\vec{n}_2$ determine directions of spin projection axes. Hence, the latter tomogram can be rewritten as

$$\omega(m_1,m_2|\vec{n}_1,\vec{n}_2)= \qquad (5)$$
$$=<m_1m_2|u_1(\vec{n}_1)\otimes u_2(\vec{n}_2)\rho u_1^\dagger(\vec{n}_1)\otimes u_2^\dagger(\vec{n}_2)|m_1m_2>.$$

The latter tomogram is the conditional probability of projections of spins m_1, m_2 on vectors $\vec{n}_1$, $\vec{n}_2$ on the Bloch sphere. Hence, (5) is the tomographic representation of the probability (1). The probability function (5) has the property of a no-signaling. Hence, the marginal probability distributions of the first and the second qubit are given by

$$\omega_1(m_1|\vec{n}_1)=\sum_{m_2=-j_2}^{j_2}\omega_1(m_1,m_2|\vec{n}_1), \qquad (6)$$
$$\omega_1(m_2|\vec{n}_2)=\sum_{m_1=-j_1}^{j_1}\omega_1(m_1,m_2|\vec{n}_2).$$

It is known that the state is called separable if and only if the density operator of the composite system ρ can be written as

$$\rho=\sum_k p_k\rho_1^{(k)}\otimes\rho_2^{(k)},\quad \sum_k p_k=1.$$

Hence, using (6) tomogram (5) can be written as

$$\omega(m_1,m_2|\vec{n}_1,\vec{n}_2)=\sum_k p_k\omega_1^{(k)}(m_1|\vec{n}_1)\omega_2^{(k)}(m_2|\vec{n}_2)$$
$$=\sum_\lambda p(\lambda)\omega_1(m_1|\vec{n}_1,\lambda)\omega_2(m_2|\vec{n}_2,\lambda).$$

We can rewrite the latter expression in the form (1), which is the LHS model in the form of the tomogram.

The tomogram can be represented using the operator $\widehat{U}(m_1,m_2,\vec{n}_1,\vec{n}_2)$ called the dequantizer. What's more, by the given spin tomogram one can reconstruct the operator of the density matrix $\widehat{\rho}$ using the operator $\widehat{D}(m_1,m_2,\vec{n}_1,\vec{n}_2)$ called the quantizer

$$\omega(m_1,m_2,\vec{n}_1,\vec{n}_2)=Tr(\widehat{\rho}\widehat{U}(m_1,m_2,\vec{n}_1,\vec{n}_2)), \qquad (7)$$
$$\widehat{\rho}=\sum_{m_1,m_2}\int\omega(m_1,m_2,\vec{n}_1,\vec{n}_2)\widehat{D}(m_1,m_2,\vec{n}_1,\vec{n}_2)d\vec{n}_1d\vec{n}_2,$$

where it holds

$$\int d\vec{n} = \int\limits_0^{2\pi} d\varphi \int\limits_0^{\pi} \sin\theta d\theta \int\limits_0^{2\pi} d\psi.$$

In [20] the dequantizer and the quantizer operators for the two-qubit state are defined as

$$\widehat{U}(m_1, m_2, \vec{n}_1, \vec{n}_2) = \widehat{U}(m_1, \vec{n}_1) \otimes \widehat{U}(m_2, \vec{n}_2) \tag{8}$$
$$= \left(\frac{1}{2}\widehat{I} + m_1 F(\varphi_1, \theta_1)\right) \otimes \left(\frac{1}{2}\widehat{I} + m_2 F(\varphi_2, \theta_2)\right),$$
$$\widehat{D}(m_1, m_2, \vec{n}_1, \vec{n}_2) = \widehat{D}(m_1, \vec{n}_1) \otimes \widehat{D}(m_2, \vec{n}_2)$$
$$= \left(\frac{1}{8\pi^2}\left(\frac{1}{2}\widehat{I} + 3m_1 F(\varphi_1, \theta_1)\right)\right) \otimes \tag{9}$$
$$\otimes \left(\frac{1}{8\pi^2}\left(\frac{1}{2}\widehat{I} + 3m_2 F(\varphi_2, \theta_2)\right)\right),$$

where $\widehat{I}$ is the 2×2 identity matrix and

$$F(\varphi, \theta) = \begin{pmatrix} \cos\theta & -e^{i\varphi}\sin\theta \\ -e^{-i\varphi}\sin\theta & -\cos\theta \end{pmatrix}.$$

Any observable A can be identified with a Hermitian operator $\widehat{A}$. In [21] the tomographic symbol $\omega_A(\mathbf{x})$ of the operator $\widehat{A}$ is determined by

$$\omega_A(\mathbf{x}) = Tr(\widehat{A}\widehat{U}(\mathbf{x})), \quad \widehat{A} = \int \omega_A(\mathbf{x})\widehat{D}(\mathbf{x})d\mathbf{x},$$

where $\widehat{U}(\mathbf{x})$, $\widehat{D}(\mathbf{x})$ are the dequantizer and quantizer operators, respectively. Using the quantizer operator as the dequantizer and vise versa the dual tomographic symbol $\omega^d_A(\mathbf{x})$ is introduced

$$\omega^d_A(\mathbf{x}) = Tr(\widehat{A}\widehat{U}'(\mathbf{x})) = Tr(\widehat{A}\widehat{D}(\mathbf{x})), \tag{10}$$
$$\widehat{A} = \int \omega^d_A(\mathbf{x})\widehat{D}'(\mathbf{x})d\mathbf{x} = \int \omega^d_A(\mathbf{x})\widehat{U}(\mathbf{x})d\mathbf{x}.$$

Using the dual tomographic symbol (11) we can write the following trace

$$Tr(\widehat{A}\widehat{\rho}) = \int \omega_A(\mathbf{x})\omega^d_\rho(\mathbf{x})d\mathbf{x}. \tag{11}$$

Hence, using (11) the quantum correlation function (3) can be rewritten in the tomographic form

$$E(\overrightarrow{k_1}, \overrightarrow{k_2}) = \int \omega_B(\mathbf{x})\omega^d_\rho(\mathbf{x})d\mathbf{x} \tag{12}$$

or in the equivalent form

$$E(\overrightarrow{k_1}, \overrightarrow{k_2}) = \int \omega_\rho(\mathbf{x})\omega^d_B(\mathbf{x})d\mathbf{x} \tag{13}$$
$$= \sum_{m_1, m_2} \int \omega_\rho(m_1, m_2, \vec{n}_1, \vec{n}_2)\omega^d_B(m_1, m_2, \vec{n}_1, \vec{n}_2)d\vec{n}_1 d\vec{n}_2,$$

where we used the notation $B = k_1\sigma \otimes k_2\sigma$. Thus, we can write any steering inequality that contains the correlation function in terms of the spin tomograms.

III. THE RELATION BETWEEN THE SINGLE QUDIT AND THE TWO-QUBIT STATES TOMOGRAMS

For the single qudit with the spin $j = 3/2$ and the density matrix ρ we can write the tomographic representation as

$$\widehat{\rho} = \sum_{m=-3/2}^{3/2} \int W(m, \vec{n})\widehat{\mathcal{D}}(m, \vec{n})d\vec{n}, \tag{14}$$
$$W(m, \vec{n}) = Tr\left(\widehat{\mathcal{U}}(m, \vec{n})\widehat{\rho}\right),$$

where $\widehat{\mathcal{U}}(m, \vec{n})$ and $\widehat{\mathcal{D}}(m, \vec{n})$ are the dequantizer and the quantizer operators of the latter state, respectively. The tomogram $W(m, \vec{n})$ is the conditional probability of projections of the spin m on vector $\vec{n}$ on the Bloch sphere. Note that the later tomogram depends on less amount of numbers comparable to the two-qubit state tomogram (5).

In [22] it was obtained that any qudit state with the spin j can be represented as

$$\widehat{\rho}^{(j)} = \sum_{m=-j}^{j} \int \frac{d\omega}{8\pi^2} W(m, \alpha, \beta)\widehat{B}^j_m(\alpha, \beta),$$

where $\widehat{B}^j_{m_1}(\alpha, \beta)$ is the quantizer operator. For the single qudit state, the latter operator is defined by the following matrix

$$B^{3/2}_m(\alpha, \beta) = B_{1m}(\alpha, \beta) + \frac{i(-1)^m}{2(m+\frac{3}{2})!(\frac{3}{2}-m)!}$$
$$\cdot \left(5mB_{2m}(\alpha, \beta) + \frac{21}{2}\sin\beta B_{3m}(\alpha, \beta)\right),$$

where we used the notations

$$B_{1m}(\alpha, \beta) = \begin{pmatrix} \frac{1}{4} + \frac{9}{10}m\cos\beta & \frac{3\sqrt{3}m}{10}\sin\beta e^{-\alpha i} & 0 & 0 \\ \frac{3\sqrt{3}m}{10}\sin\beta e^{\alpha i} & \frac{1}{4} + \frac{3}{10}m\cos\beta & \frac{63m}{105}\sin\beta e^{\alpha i} & 0 \\ 0 & \frac{63m}{105}\sin\beta e^{\alpha i} & \frac{1}{4} - \frac{3}{10}m\cos\beta & \frac{3\sqrt{3}m}{10}\sin\beta e^{-\alpha i} \\ 0 & 0 & \frac{3\sqrt{3}m}{10}\sin\beta e^{\alpha i} & \frac{1}{4} - \frac{9}{10}m\cos\beta \end{pmatrix},$$

$$B_{2m}(\alpha,\beta)=\begin{pmatrix} 3\cos^2\beta-1 & \sqrt{3}e^{-\alpha i}\sin 2\beta & \sqrt{3}\sin^2\beta e^{-2\alpha i} & 0 \\ \sqrt{3}\sin 2\beta e^{ai} & -3\cos^2\beta+1 & 0 & -\sqrt{3}\sin^2\beta e^{-2\alpha i} \\ \sqrt{3}\sin^2\beta e^{2ai} & 0 & -3\cos^2\beta+1 & -\sqrt{3}\sin 2\beta e^{-\alpha i} \\ 0 & -\sqrt{3}\sin^2\beta e^{2ai} & -\sqrt{3}\sin 2\beta e^{ai} & 3\cos^2\beta-1 \end{pmatrix},$$

$$B_{3m}(\alpha,\beta)=\begin{pmatrix} \frac{\cos\beta}{\sin\beta}(\cos^2\beta-\frac{3}{5}) & \sqrt{3}\left(\cos^2\beta-\frac{1}{5}\right)e^{-\alpha i} & \sqrt{3}\sin\beta\cos\beta e^{-2\alpha i} & \sin^2\beta e^{-3\alpha i} \\ \sqrt{3}\left(\cos^2\beta-\frac{1}{5}\right)e^{\alpha i} & 3\frac{\cos\beta}{\sin\beta}(\cos^2\beta-\frac{3}{5}) & 3(\frac{1}{5}-\cos^2\beta)e^{-\alpha i} & -\sqrt{3}\sin\beta\cos\beta e^{-2\alpha i} \\ \sqrt{3}\sin\beta\cos\beta e^{2\alpha i} & 3(\frac{1}{5}-\cos^2\beta)e^{\alpha i} & -3\frac{\cos\beta}{\sin\beta}(\cos^2\beta-\frac{3}{5}) & \sqrt{3}\left(\cos^2\beta-\frac{1}{5}\right)e^{-\alpha i} \\ \sin^2\beta e^{3\alpha i} & \sqrt{3}\sin\beta\cos\beta e^{2\alpha i} & \sqrt{3}(\cos^2\beta-\frac{1}{5})e^{\alpha i} & -\frac{\cos\beta}{\sin\beta}(\cos^2\beta-\frac{3}{5}) \end{pmatrix}.$$

We can use the latter operator as the quantizer for the single qudit system, i.e $\widehat{\mathcal{D}}(m,\alpha,\beta)=\frac{1}{8\pi^2}\widehat{B}^{3/2}_m(\alpha,\beta)$. To write the rotation matrix $U^{(3/2)}(\alpha,\beta,\gamma)$ for the single qudit state we can use the Wigner's D-function

$$D^{(j)}_{m',m}(\alpha,\beta,\gamma)=e^{im'\gamma}d^{(j)}_{m',m}(\beta)e^{im'\alpha},$$

where it holds

$$d^{(j)}_{m',m}(\beta)=\left(\frac{(j+m')!(j-m')!}{(j+m)!(j-m)!}\right)^{1/2}\cos\left(\frac{\beta}{2}\right)^{m'+m} \cdot \sin\left(\frac{\beta}{2}\right)^{m'-m}P^{(m'-m,m'+m)}_{j-m'}(\cos\beta), \tag{15}$$

and $P^{(m'-m,m'+m)}_{j-m'}(\cos\beta)$ denote the Jacobi polynomials [13]. Hence, the rotation matrix for the single qudit state $U^{(3/2)}(\alpha,\beta,\gamma)$ has elements $U^{(3/2)}_{m',m}(\alpha,\beta,\gamma)=D^{(3/2)}_{m',m}(\alpha,\beta,\gamma)$. Using the latter rotation matrix the dequantizer operator can be defined by the following matrices

$$\mathcal{U}(m,\alpha,\beta)=U^{(3/2)\dagger}(\alpha,\beta,\gamma)|m><m|U^{(3/2)}(\alpha,\beta,\gamma).$$

To find the relation between the two-qubit system and the single qudit system tomograms we substitute (14) into (7). We get

$$\begin{aligned}\omega(m_1,m_2,\vec{n}_1,\vec{n}_2)= & \\ =\int TrW(m,\vec{n})\widehat{\mathcal{D}}(m,\vec{n})\widehat{U}(m_1,m_2,\vec{n}_1,\vec{n}_2)d\vec{n} & \\ =\int W(m,\vec{n})K_{12}d\vec{n}, &\end{aligned} \tag{16}$$

where the notation $K_{12}\equiv K_{12}(m_1,m_2,m,\vec{n}_1,\vec{n}_2,\vec{n})=Tr\widehat{\mathcal{D}}(m,\vec{n})\widehat{U}(m_1,m_2,\vec{n}_1,\vec{n}_2)$ is introduced. We call it the kernel function. Using (8) and (15) we can write the latter kernel in the explicit form

$$\begin{aligned}K_{12}(m,m_1,m_2,\alpha,\beta,\theta_1,\theta_2,\varphi_1,\varphi_2)=\frac{1}{4}+\frac{3m}{5}\Bigg(&\cos\beta(2m_1\cos\theta_1+m_2\cos\theta_2)\\ &+m_2\sin\beta\cos\alpha\sin\theta_2 e^{i\varphi_2}\left(-\sqrt{3}+2m_1\sin\theta_1 e^{i\varphi_1}\right)\Bigg)-\frac{(-1)^m i}{(m+3/2)!(3/2-m)!}\\ &\cdot\Bigg(21\cos\beta\left(-\frac{3}{5}+cos^2\beta\right)\left(\frac{m_1}{2}\cos\theta_1-m_2\cos\theta_2\right)+10mm_1m_2\cos\theta_1\cos\theta_2\left(1-3\cos^2\beta\right)\\ &+\sqrt{3}m_2\Bigg(\frac{21}{2}\sin\beta\sin\theta_2 e^{i\varphi_2}\cos\alpha\left(\cos^2\beta-\frac{1}{5}\right)+21m_1\cos\beta\sin^2\beta\cos\theta_2\sin\theta_1 e^{i\varphi_1}\cos 2\alpha\\ &+10mm_1\sin^2\beta\left(e^{i\varphi_1}\cos\theta_2\sin\theta_1\cos 2\alpha+4e^{i\varphi_2}\cos\theta_1\sin\theta_2\cos\alpha\right)\Bigg)\Bigg)\\ &+\frac{21m_1m_2}{2}\sin\beta\sin\theta_1\sin\theta_2 e^{i\varphi_1}e^{i\varphi_2}\left(-\frac{3}{5}\cos\alpha+3\cos^2\beta\cos\alpha-\sin^2\beta\cos 3\alpha\right)\end{aligned} \tag{17}$$

The kernel depends on three quantum numbers and six angles. Similarly we can write the inverse transformation as

$$W(m,\vec{n}) = \int \omega(m_1,m_2,\vec{n}_1,\vec{n}_2)K_{21}d\vec{n}_1 d\vec{n}_2,$$

where the kernel is $K_{21} \equiv K_{21}(m_1,m_2,m,\vec{n}_1,\vec{n}_2,\vec{n}) = Tr\widehat{D}(m_1,m_2,\vec{n}_1,\vec{n}_2)\widehat{\mathcal{U}}(m,\vec{n})$. For the dual tomographic symbols (11) it is easy to see that

$$\omega^d(m_1,m_2,\vec{n}_1,\vec{n}_2) = \int W^d(m,\vec{n})K^d_{12}d\vec{n},$$
$$W^d(m,\vec{n}) = \int \omega^d(m_1,m_2,\vec{n}_1,\vec{n}_2)K^d_{21}d\vec{n}_1 d\vec{n}_2,$$

where the kernels are $K^d_{12} = K_{21}$ and $K^d_{21} = K_{12}$. The latter four kernels describe the connection between the tomograms and the dual tomographic symbols for the bipartite and single qudit states.

A. The Example

Let us have the quantum state described by the 4×4 Werner density matrix

$$\rho^W = \begin{pmatrix} \frac{1+p}{4} & 0 & 0 & \frac{p}{2} \\ 0 & \frac{1-p}{4} & 0 & 0 \\ 0 & 0 & \frac{1-p}{4} & 0 \\ \frac{p}{2} & 0 & 0 & \frac{1+p}{4} \end{pmatrix}, \qquad (18)$$

where the parameter p satisfies the inequality $-\frac{1}{3} \le p \le 1$. The parameter domain $\frac{1}{3} < p \le 1$ corresponds to the entangled state. If matrix (18) describes the single qudit state with the spin $j = 3/2$ the tomogram (14) has the following elements

$$\begin{aligned}
W\left(-\frac{3}{2},\alpha,\beta\right) &= \frac{p}{16} + \frac{3p}{16}\cos 2\beta - \frac{3p}{32}\sin\beta\cos 3\alpha + \frac{p}{32}\cos 3\alpha \sin 3\beta + \frac{1}{4}\\
W\left(-\frac{1}{2},\alpha,\beta\right) &= \frac{3p}{16}(2\sin^2\beta - 1) - \frac{p}{16} + \frac{3p}{32}\sin 3\beta(2\sin\frac{3\alpha}{2}^2 - 1) - \frac{9p}{32}\sin\beta(2\sin\frac{3\alpha}{2}^2 - 1) + \frac{1}{4}\\
W\left(\frac{1}{2},\alpha,\beta\right) &= \frac{3p}{16}(2\sin^2\beta - 1) - \frac{p}{16} - \frac{3p}{32}\sin 3\beta(2\sin\frac{3\alpha}{2}^2 - 1) + \frac{9p}{32}\sin\beta(2\sin\frac{3\alpha}{2}^2 - 1) + \frac{1}{4}\\
W\left(\frac{3}{2},\alpha,\beta\right) &= \frac{p}{16} + \frac{3p}{16}\cos 2\beta + \frac{3p}{32}\sin\beta\cos 3\alpha - \frac{p}{32}\cos 3\alpha \sin 3\beta + \frac{1}{4}.
\end{aligned}$$

From the other hand, the density matrix (18) can describe the two-qubit state. Hence, substituting the latter tomogram and the kernel (17) into (5) we can get the tomogram for the two-qubit Werner density matrix

$$\begin{aligned}
\omega(m_1,m_2,\theta_1,\theta_2,\phi_1,\phi_2) &= 2\pi\sum_{m=-3/2}^{3/2}\int_0^{2\pi}\int_0^{\pi}\sin\beta W(m,\alpha,\beta)K_{12}(m,m_1,m_2,\alpha,\beta,\theta_1,\theta_2,\phi_1,\phi_2)d\beta d\alpha\\
&= \frac{1}{4} + pm_1m_2\left(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2 e^{i\phi_1}e^{i\phi_2}\right)
\end{aligned}$$

which coincides with (7).

IV. THE SINGLE QUDIT STEERING IN THE SPIN TOMOGRAPHIC REPRESENTATION

In Sec. II the correlation function $E(\overrightarrow{k}_1,\overrightarrow{k}_2)$ is obtained by measuring the quantum observable $\widehat{O} = \overrightarrow{k_1}\cdot\overrightarrow{\sigma}\otimes\overrightarrow{k_2}\cdot\overrightarrow{\sigma}$. The 4×4 matrix form of this observable reads

$$\begin{pmatrix} k_{1z}k_{2z} & k_{1z}(k_{2x}-k_{2y}i) & k_{2z}(k_{1x}-k_{1y}i) & (k_{1x}-k_{1y}i)(k_{2x}-k_{2y}i) \\ k_{1z}(k_{2x}+k_{2y}i) & -k_{1z}k_{2z} & (k_{1x}-k_{1y}i)(k_{2x}+k_{2y}i) & -k_{2z}(k_{1x}-k_{1y}i) \\ k_{2z}(k_{1x}+k_{1y}i) & (k_{1x}+k_{1y}i)(k_{2x}-k_{2y}i) & -k_{1z}k_{2z} & -k_{1z}(k_{2x}-k_{2y}i) \\ (k_{1x}+k_{1y}i)(k_{2x}+k_{2y}i) & -k_{2z}(k_{1x}+k_{1y}i) & -k_{1z}(k_{2x}+k_{2y}i) & k_{1z}k_{2z} \end{pmatrix}$$

For the two-qubit system the measuring of observable $\widehat{O}$ is connected with the measuring spin projection of the first spin on the direction given by the vector $\overrightarrow{k_1}$ and for the second spin projection on the direction given by the vector $\overrightarrow{k_2}$. In fact, this observable is determined by two commuting observables $\widehat{O}_1 = \overrightarrow{k_1} \cdot \overrightarrow{\sigma} \otimes \widehat{1}_2$ and $\widehat{O}_2 = \widehat{1}_1 \otimes \overrightarrow{k_2} \cdot \overrightarrow{\sigma}$ which have matrix representations

$$O_1 = \begin{pmatrix} k_{1z} & 0 & k_{1x}-k_{1y}i & 0 \\ 0 & k_{1z} & 0 & k_{1x}-k_{1y}i \\ k_{1x}+k_{1y}i & 0 & -k_{1z} & 0 \\ 0 & k_{1x}+k_{1y}i & 0 & -k_{1z} \end{pmatrix},$$

$$O_2 = \begin{pmatrix} k_{2z} & k_{2x}-k_{2y}i & 0 & 0 \\ k_{2x}+k_{2y}i & -k_{2z} & 0 & 0 \\ 0 & 0 & k_{2z} & k_{2x}-k_{2y}i \\ 0 & 0 & k_{2x}-k_{2y}i & -k_{2z} \end{pmatrix}.$$

It is obvious that $\widehat{O} = \widehat{O}_1 \cdot \widehat{O}_2$ holds. In this sense the commuting observables $\widehat{O}_1$ and $\widehat{O}_2$ can be measured simultaneously and the measurement of the correlation function $E(\overrightarrow{k}_1, \overrightarrow{k}_2)$ or the measurement of the observable $\widehat{O}$ is reduced to the measuring the observables $\widehat{O}_1$ and $\widehat{O}_2$ for the two qubits or the two-level atom.

The physical meaning of the correlation function is the following. The correlation function for the two qubits is used to discuss the Bells inequalities [23, 24] and their violation is detected, e.g., in [25]. In the case of the Bell inequalities the correlation function $E(\overrightarrow{k}_1, \overrightarrow{k}_2)$ is considered for four pairs of directions $\overrightarrow{k}_1$ and $\overrightarrow{k}_1$, namely $(\overrightarrow{k}_1, \overrightarrow{k}_2) = \{(\overrightarrow{a}, \overrightarrow{b}); (\overrightarrow{a}, \overrightarrow{c}); (\overrightarrow{d}, \overrightarrow{b}); (\overrightarrow{d}, \overrightarrow{c})\}$. It is known that for the two qubits the Bell inequality reads as

$$|E(\overrightarrow{a}, \overrightarrow{b}) + E(\overrightarrow{a}, \overrightarrow{c}) + E(\overrightarrow{d}, \overrightarrow{b}) - E(\overrightarrow{d}, \overrightarrow{c})| \leq 2$$

for separable states and

$$|E(\overrightarrow{a}, \overrightarrow{b}) + E(\overrightarrow{a}, \overrightarrow{c}) + E(\overrightarrow{d}, \overrightarrow{b}) - E(\overrightarrow{d}, \overrightarrow{c})| \leq 2\sqrt{2}$$

holds for the entangled states. In [10] the steering inequality for the two-qubit state is based on the maxima of the correlation function (3) given in the following form

$$E(\overrightarrow{k}_1, \overrightarrow{k}_2) = \sum_{i,j=1}^{3} T_{ij} k_{1i} k_{2j},$$

where T_{ij} are components of the correlation matrix. If the bipartite state is non-steerable, then the following inequality is fulfilled

$$\max_{\overrightarrow{k}_1, \overrightarrow{k}_2} (E(\overrightarrow{k}_1, \overrightarrow{k}_2)) \geq \frac{2}{3} \sum_{i,j=1}^{3} T_{ij}. \qquad (19)$$

One can see that the Bell inequalities and the steering phenomenon reflect different aspects of the quantum correlations in the two-qubit system.

To formulate the steering phenomenon for the four level atom or for the spin $j = 3/2$ states we apply the same formalism used for two-qubit system states. We introduce two observables $\widehat{\mathcal{O}}_1$ and $\widehat{\mathcal{O}}_2$ (i.e Hermitian operators) for the single qudit system. The observables $\widehat{\mathcal{O}}_1$ and $\widehat{\mathcal{O}}_2$ (operators) act in the Hilbert space of the states of the four-level atom ($j = 3/2$ qudit). The matrix form of the observables $\widehat{\mathcal{O}}_1$, $\widehat{\mathcal{O}}_2$ in the basis $|3/2, 3/2>$, $|3/2, 1/2>$, $|3/2, -1/2>$, $|3/2, -3/2>$ of the spin system is identical to $\widehat{O}_1$ and $\widehat{O}_2$.

Then we introduce the observable $\widehat{\mathcal{O}} = \widehat{\mathcal{O}}_1 \cdot \widehat{\mathcal{O}}_2$ which is the Hermitian operator acting in the Hilbert space $\mathcal{H}$ which is not considered as a product, i.e. $\mathcal{H} \neq \mathcal{H}_1 \otimes \mathcal{H}_2$. This product form was used for the system of the two qubits. Nevertheless due to postulates of qunatum mechanics any observable (the Hermitian operator) can be measured. We suggest to introduce the steering notion for the single qudit (e.g., the spin $j = 3/2$) to be based on the measuring of the correlation function $\mathcal{E}(\overrightarrow{k}_1, \overrightarrow{k}_2)$ which corresponds to the measuring of the observable $\widehat{\mathcal{O}}$ in states of the single qudit. It is the main tool to translate the steering properties known for the composite systems (two qubits) to the noncomposite systems as well as a single qudit (four-level atom).

Hence, similarly to (12) we can write the correlation function of the single qudit state as

$$\mathcal{E}(\overrightarrow{k}_1, \overrightarrow{k}_2) = \int W_{k_1\sigma \otimes k_2\sigma}(\mathbf{y}) W_\rho^d(\mathbf{y}) d\mathbf{y}, \qquad (20)$$

where the tomograms are defined by (14).

Using the intertwining kernel (17) we can deduce that the correlation functions (12) and (20) are mathematically completely equivalent.

$$E(\overrightarrow{k}_1,\overrightarrow{k}_2)=\sum_{m_1,m_2=-1/2}^{1/2}\int\int d\vec{n_1}d\vec{n_2}\sum_{m=-3/2}^{3/2}\int W_\rho(m,\vec{n})W^d_{k_1\sigma\otimes k_2\sigma}(m,\vec{n})K_{12}K_{21}d\vec{n}=\mathcal{E}(\overrightarrow{k}_1,\overrightarrow{k}_2). \tag{21}$$

Here we use that $K_{12}K_{21}=1$ holds.

A. Means of the Correlation Function

Let the single qudit state with the spin $j=(N-1)/2$ be given by the $N\times N$ density matrix $\rho_{ss'}$, $s,s'=1,2\ldots,N$. For the $N=n_1n_2$, the latter density matrix can be interpreted as the density matrix of the two qudits with the spins $j_1=(n_1-1)/2$ and $j_2=(n_2-1)/2$. It is obvious, that at $N=n_1n_2n_3$ the matrix $\rho_{ss'}$ can also be the density matrix of the three qudits with the spins $j_i=(n_i-1)/2, i=1,2,3$. It is easy to see that the latter interpretations can be provided for any l-qudits system. To this end the invertible map of the matrix indices can be used, i.e. for the two qudits $s\leftrightarrow j,k$, $s'\leftrightarrow j',k'$ ($s=s(j,k)$, $s'=s'(j',k')$) or $s\leftrightarrow j,k,l$, $s'\leftrightarrow j',k',l'$ ($s=s(j,k,l)$, $s'=s'(j',k',l')$) for the three qudits system case. The latter approach was used to consider the correlations in the single qudit system as the correlations in artificial multi qudit system in [14]. The similar interpretation was done also for the matrices of the observables $F_{ss'}$ corresponding to the operators $\widehat{F}$ on the Hilbert space $\mathcal{H}$. The matrices of observables were written in the following forms

$$\begin{aligned}F_{ss'}&=F_{s(j,k)s'(j',k')}\equiv F_{jk,j'k'},\\ F_{ss'}&=F_{s(j,k,l)s'(j',k',l')}\equiv F_{jkl,j'k'l'}\end{aligned} \tag{22}$$

and the density matrices of the states are

$$\begin{aligned}\rho_{ss'}&=\rho_{s(j,k)s'(j',k')}\equiv \rho_{jk,j'k'},\\ \rho_{ss'}&=\rho_{s(j,k,l)s'(j',k',l')}\equiv \rho_{jkl,j'k'l'}.\end{aligned} \tag{23}$$

Thus, the quantum states and the quantum observables given by the density operator $\widehat{\rho}$ and the observable operator $\widehat{F}$ can be associated with (22) and (23) given in the basis $|s>$. However, the basis can be considered as $|s>=|s(j,k)>=|j>|k>$ on the Hilbers space given by the tensor product of the two Hilbert spaces $\widehat{\mathcal{H}}=\widehat{\mathcal{H}}_1\otimes\widehat{\mathcal{H}}_2$. The density operator and the observable operator can be rewritten in the terms of the indices $jk,j'k'$, $j,j'=1,2,\ldots,n_1$, $k,k'=1,2,\ldots,n_2$ as

$$\begin{aligned}\rho_{jk,j'k'}&=<s(j,k)|\widehat{\rho}|s'(j',k')>,\\ F_{jk,j'k'}&=<s(j,k)|\widehat{F}|s'(j',k')>.\end{aligned} \tag{24}$$

Let us take the observable operator as $\widehat{F}=\widehat{F}_1\otimes\widehat{F}_2$, where the operators $\widehat{F}_1$, $\widehat{F}_2$ are acting on the Hilbert spaces $\widehat{\mathcal{H}}_1$ and $\widehat{\mathcal{H}}_2$, respectively. The mean value of the observable $F_{ss'}$ is

$$<\widehat{F}>=\sum_{j,j'=1}^{n_1}\sum_{k,k'=1}^{n_2}F_{1_{jj'}}F_{2_{kk'}}\rho_{jk,j'k'}.$$

Introducing the two commuting observables $\widehat{f}_1=\widehat{F}_1\otimes\widehat{1}_m$ and $\widehat{f}_2=\widehat{1}_n\otimes\widehat{F}_2$ we can write the mean value of $\widehat{F}$ in form of the correlation function of the observables $\widehat{f}_1$ and $\widehat{f}_2$, i.e. $<\widehat{F}>=<\widehat{f}_1\widehat{f}_2>$. The latter relation can be generalized to the case of l commuting observables $\widehat{f}_i, i=1,2,\ldots,l$, $<\widehat{F}>=<\widehat{f}_1\widehat{f}_2\cdots\widehat{f}_l>$. Hence, the mean value of the observable $\widehat{F}$ can be interpreted as the correlation function of the observables $f_i, i=1,2,\ldots l$. The interpretation can be the existence of the hidden correlations in the single qudit system.

That means that the quantum correlations known for the observables associated with the subsystems are also valid in the single qudit systems like the quantum roulette and the quantum compass. As the quantum roulette or the quatum compass we mean the single qudit system realized by the artificial four-level atom presented by the energy levels of Josephson junctions in the experiment with the superconducting circuits [19, 26–29]. The observables which are the analogs of the spins in the two-qubit systems for the four-level artificial atoms and the correlations of these observables can be in principle measured in experiments in measuring of the population in the energy levels of the superconducting circuits. In view of this the steering phenomenon can be detected in the latter systems.

B. The Example

If the Werner density matrix (18) describes the two-qubit state then the correlation function (12) is

$$E(\overrightarrow{k_1},\overrightarrow{k_2})=k_{1z}k_{2z}p. \tag{25}$$

The correlation tensor (19) is the diagonal matrix with entries $p(1-1\quad 1)$ and the maximum value of the correlation function (25) is equal to p in the domain $0<p<1/3$ and to $-p$ in $-1/3<p\leq 0$. Hence, the inequality (19) is fulfilled if $0<p<1/2$ holds. Since we are interested only in the entangled states, the parameter domain $1/3<p<1/2$ corresponds to the steerable Werner state.

If the Werner density matrix (18) describes the single qudit $j=3/2$ state it is straightforward to verify that the correlation function (12) has the form (25) that coincides with (21).

V. SUMMARY

To resume we formulate the main result of our work. We have shown that the quantum correlations reflected by the phenomenon of the quantum steering available in the two-qubit system take place also in the single qudit $j = 3/2$. To this end we used the special mapping of the states of the single qudit systems to the states of the artificial multi qudit systems. We demonstrate the inequalities for the correlation function detecting the presence of the steering not only for the two-qubit states but also for the single qudit $j = 3/2$ state. The physical meaning of these hidden correlations is different from the case of two qubits correlations. The observables to be measured for the obtained correlation being mathematically completely equivalent to observables measured in the experiment with two qubits are different for the single qudit. The results are obtained by using the tomographic probability representation of the quantum states and the intertwining kernel related to the two-qubit state tomogram and to the qudit $j = 3/2$ tomogram is explicitly calculated. The results are illustrated by the example of the Werner density matrix that can describe the two-qubit and the single qudit states. The extension of the steering consideration for the other single qudits will be done in our future work.

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